

Phase Structure of Four-dimensional Simplicial Quantum Gravity with a $U(1)$ Gauge Field

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Abstract

The phase structure of four-dimensional simplicial quantum gravity coupled to $U(1)$ gauge fields has been studied using Monte-Carlo simulations. The smooth phase is found in the intermediate region between the crumpled phase and the branched polymer phase. This new phase has a negative string susceptibility exponent, even if the number of vector fields (N_V) is 1. The phase transition between the crumpled phase and the smooth phase has been studied by a finite size scaling method. From the numerical results, we expect that this model (coupled to one gauge field) has a higher order phase transition than first order, which means the possibility to take the continuum limit at the critical point. Furthermore, we consider a modification of the balls-in-boxes model for a clear understanding of the relation between the numerical results and the analytical one.

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1 Introduction

To formulate a theory of quantum gravity in four dimensions, many approaches have been tried. One of them is a numerical approach with the method of dynamical triangulation. In two dimensions, the quantum gravity can be quantized for a central charge $c < 1$. The method of dynamical triangulation has generally been considered to be a correct discretized model, and has given consistent results with the analytical approach: for example, the MINBU analysis [1,2] and the loop length distributions [3,4]. Two-dimensional quantum gravity has generally been regarded as being a toy model of four-dimensional quantum gravity. Recently, a numerical approach with the method of dynamical triangulation in analogy with a two-dimensional model has been studied. In the four-dimensional case, it has been expected that the phase transition point between the strong coupling phase and the weak coupling phase is statistically the second phase transition point. Moreover, this point is recognized as the ultraviolet (UV) fixed point of the quantum theory of gravity.

From numerical results, in four-dimensional pure gravity, it is known that there are two distinct phases [5–10]. For small values of the bare gravitational coupling constant, the system is in the so-called elongated phase, which has the characteristics of a branched polymer phase. For large values of the bare gravitational coupling constant, it is in the so-called crumpled phase. Numerically, the phase transition between the two phases has actually been shown to be of first order [11,12]. Therefore, it is difficult to construct a continuum theory. In other words, It is difficult to define the quantum theory of gravity on a four-dimensional triangle (4-simplex) as a simple application of the two-dimensional lattice model.

Our next step is to investigate the possibilities to extend the model of four-dimensional quantum gravity. We have three motivations: (1) a modified model [8] in three-dimensional case, which suppresses the vertex order concentration (VOC) as the singular sub-simplex, and which can be changed of the phase structures, (2) the property of quantum field theory with background metric independence [13–15], that the manifold may be made stable by adding matter fields and (3) the balls-in-boxes model, that gives a scenario for a phase transition in four-dimensional simplicial quantum gravity [16,17], and that shows possibilities to change the phase structure with some modifications of the model [18]. For the possibility of a continuum theory, at least, we have to find a new phase structure that has a second order phase transition point. One modification is to introduce gauge matter fields [19].

Recently, the phase structure with vector fields has been studied numerically [21,20]. In the case of a model with vector fields, the phase structure is changed drastically and the intermediate phase, the so-called smooth phase, has been observed between the crumpled phase and the elongated phase. In this region, the string susceptibility exponents (γ_{st}) have negative values. They show that this region has a fractal property, and may be smooth compared with the branched polymer region. We thus expect the possibility of a continuum limit at the critical point between the crumpled phase and the smooth phase. In order to investigate the nature of the phase transition, we measure the critical exponent as the finite size scaling, and also study the scaling property of the mother boundary in analogy with the the two-dimensional case, and we expect that the scaling structure also appears in the boundary in the four-dimensional case.

This paper is organized as follows. In Section 2, we discuss the model of four-dimensional dynamical triangulation with some vector fields. In Section 3, we show our numerical results concerning measurements of the string susceptibility exponents (γ_{st}) and a schematic phase diagram. We thus discuss the statistical property of the phase transition between the crumpled phase and the smooth phase in the case of four-dimensional simplicial quantum gravity coupled

to one gauge field ($N_V = 1$). Furthermore, in section 4, we discuss the scaling property near to the critical point. In section 5, we discuss a scenario for the phase structure and the phase transition in four-dimensional simplicial quantum gravity coupled to matter fields. Finally, we discuss the possibility of a continuum limit of four-dimensional simplicial quantum gravity in this article.

2 Model

It is still not known how to provide a constructive definition of four-dimensional quantum gravity. We have considered a discretized random closed manifold in analogy with the two-dimensional case. We numerically evaluated the Euclidean path integral with the technique of dynamical triangulation, which gives a discrete summing over all possible connections of lattices that may replace the integral over diffeomorphism inequivalent metrics. Then, we naturally considered the Euclidean Einstein-Hilbert action coupled to N_V copies of $U(1)$ vector fields and its discretized model with 4-simplices. The total action is $S = S_{EH} + S_{pl}$. We use the Einstein-Hilbert term for gravity,

$$S_{EH}[\Lambda, G] = \int d^4x \sqrt{g} \left(\Lambda - \frac{1}{G} R \right), \quad (1)$$

where Λ is the cosmological constant and G is Newton's constant. We use the discretized action for gravity,

$$\begin{aligned} S_{EH}[\kappa_2, \kappa_4] &= \kappa_4 N_4 - \kappa_2 N_2 \\ &= -\frac{2\pi}{G} N_2 + \left(\Lambda' + \frac{10}{G} \cos^{-1} \left(\frac{1}{4} \right) \right), \end{aligned} \quad (2)$$

where $\kappa_2 \sim 1/G$, κ_4 is related to $\Lambda' = c\Lambda$ (c is the unit volume) and N_i is the number of i -simplices. We use the plaquette action for $U(1)$ gauge fields [19],

$$S_{pl} = \sum_{t_{ijk}} o(t_{ijk}) [A(l_{ij}) + A(l_{jk}) + A(l_{ki})]^2, \quad (3)$$

where l_{ij} denotes a link between vertices i and j , t_{ijk} denotes a triangle with vertices i , j and k and $o(t_{ijk})$ denotes the number of 4-simplices sharing triangle t_{ijk} . We consider the $U(1)$ gauge field $A(l_{ij}) = -A(l_{ji})$ on a link l_{ij} .

We consider that a partition function of gravity with N_V copies of $U(1)$ gauge fields is

$$Z(\kappa_2, \kappa_4, N_V) = \sum_{N_4} e^{-\kappa_4 N_4} \sum_{t(2D) \in T(4D)} e^{\kappa_2 N_2} \prod_{N_V} \int \prod_{l \in t(2D)} dA(l) e^{-S_{pl}}. \quad (4)$$

We sum over all four-dimensional simplicial triangulation $T(4D)$ in order to carry out a path integral over the metric, where we fix the topology with S^4 . As is well known, we must add a small correction term (δS) [6] to the lattice for fluctuations of volume,

$$\delta S = \delta \kappa_2 (N_4 - N_4^{target})^2, \quad (5)$$

where $\delta \kappa_2$ is adjusted with an appropriate choice; we use $\delta \kappa_2 = 0.00025$.

Near to the critical point, it is expected that the partition function (eq.(4)) behaves as

$$\begin{aligned} Z(k_2 \sim k_2^c) &\sim \sum_{T(4D)} N_4^{\gamma_{st}-3} e^{\mu N_2} \\ &\sim 1/|N_4 - N_4^{target}|^{3-\gamma_{st}}, \end{aligned} \quad (6)$$

where γ_{st} is the string susceptibility exponent related to the entropy of the manifold. In the situation; $\gamma_{st} > 0$, the manifold grows to the spiky configuration. Conversely, in $\gamma_{st} < 0$, the surfaces grow to smooth structures.

Using Monte-Carlo simulations, we evaluated the partition function (eq.(4)). We followed the way of updating the configuration after ref. [19]. This is that the gauge fields renew the sequence of the (p, q) -move according to the weight of the Boltzmann factor. The action was checked by the Metropolis methods. As we held the same conditions with Bilke et al., we performed a geometry update, (p, q) -move, updating the gauge fields by the heat-bath sweep and the over-relaxation sweep. An over-relaxation sweep was introduced for the convergence,

$$A_{ij} \rightarrow A_{ij} - 2\bar{A}_{ij}. \quad (7)$$

The measurement chance ($N_4 = N_4^{target}$) came at intervals of about 100 sweeps, where we counted “1 sweep” as the flow of “heat-bath sweep \rightarrow (p, q) -move \rightarrow over-relaxation sweep”. The measurement processes and updating processes are the same as the pure gravity case. However, since the case of adding vector fields costs more CPU time, we arranged for suitable compute time.

3 Phase diagram with vector fields

In order to investigate the phase structure of simplicial quantum gravity, first we calculated the string susceptibility exponent (γ_{st}). This exponent is defined by the asymptotic form of the partition function (eq.(6)). It is known that the case of γ_{st} corresponds to the dominance of the branched polymer [22]. We measure the string susceptibility exponent (γ_{st}) with the method of the MINBU (Minimum Necked Baby Universe) analysis [1, 2, 23]. It is a powerful exponent for probing the property of quantum geometry, and is easily measured. We can count a baby universe which is connected to the mother universe via a minimal neck. The distribution function for the MINBU analysis with the size B can be written as

$$\begin{aligned} n_A &= \frac{3(A - B + 1)(B + 1)Z[A - B + 1]Z[B + 1]}{Z[A]} \\ &\sim c \left\{ (B + 1) \left(1 - \frac{B - 1}{A} \right) \right\}^{\gamma_{st}-2}, \end{aligned} \quad (8)$$

where we use the asymptotic form of the partition function, $Z[A] \sim A^{\gamma_{st}-3} e^{\mu A}$, where A denotes the volume of the mother universe.

In the pure gravity case, a measurement of the string susceptibility exponent is discussed in ref. [23]. From the numerical results, the string susceptibility exponent supports the idea that the pure case of simplicial quantum gravity has two distinct phases. What is important in the $N_V = 1$ case is that the usual phase transition point (κ_2^c) is different from another transition point (κ_2^o), which separates the $\gamma_{st} < 0$ region from the $\gamma_{st} > 0$ region, and γ_{st} becomes negative on the phase transition point (κ_2^c). This fact leads to the definition of a new smooth phase.

This phase is defined by an intermediate region between these two transition points (κ_2^c and κ_2^o). In the pure gravity case, it is clear that $\kappa_2^c \approx \kappa_2^o$, and thus there is no evidence for the existence of a new smooth phase. On the other hand, in the case of $N_V = 1$ with $N_4 = 16K$, we observe the $\gamma_{st} < 0$ region beyond the usual phase transition point (κ_2^c). We also observe a very obscure transition from $\gamma_{st} < 0$ to $\gamma_{st} > 0$ at κ_2^o .

We give the numerical results at Table.1. In Fig.2, we plot γ_{st} for various numbers of gauge

Table 1: Measurement results of γ_{st} at $N_4 = 16K$.

κ_2	$N_V = 0$	$N_V = 1$	$N_V = 2$
1.0	-0.43(5)	-2.4(3)	-3.9(3)
1.5	0.46(2)	-0.87(9)	-2.4(2)
2.0	0.53(1)	-0.07(3)	-1.05(9)
2.5	0.59(2)	0.136(4)	-0.45(5)
3.0	0.67(1)	0.123(3)	0.01(4)
3.5	-	0.188(7)	0.15(8)

fields versus κ_2 with volume $N_4 = 16K$. In the case of adding vector fields, we can find that the usual phase-transition point (κ_2^c) is different from another transition point (κ_2^o) which separates the $\gamma_{st} < 0$ region from the $\gamma_{st} > 0$ region and γ_{st} becomes negative at the phase-transition point (κ_2^c). This fact leads to the existence of a new phase. We thus call this intermediate region the smooth phase between these two transition points (κ_2^c and κ_2^o).

We can consider that simplicial quantum gravity coupled to vector fields has three phases [21, 20, 24]. In Fig.2, we show a schematic phase diagram. We have three phases in this parameter space: a crumpled phase, a smooth phase (shaded portion) and a branched polymer phase. Furthermore, the smooth phase expands with adding more vector fields. We show the discontinuous phase transition as the thin line and the smooth phase transition as the thick line. In the case of adding the vector fields, there are two separate phase transition lines: the usual phase transition line (κ_2^c) and an obscure phase transition line (κ_2^o). The obscure transition at κ_2^o has been shown to be third order or cross over [16], which is very similar to the $c = 1$ barrier in two-dimensional quantum gravity [20]. In two dimensions the $c = 1$ barrier is well-known as an obscure transition from the fractal phase ($c \leq 1$ and $\gamma_{st}^{2D} < 0$) to the branched polymer phase ($c > 1$ and $\gamma_{st}^{2D} > 0$). We consider that the obscure phase transition point may be a threshold point, like the $c = 1$ case in two dimensions. From reports of Antoniadis et al. [13], the quantum field theory of gravity with conformal invariance has a central charge Q^2 , which has a threshold value, $Q^2 \approx 8$. We expect that the obscure phase transition point that is defined by the threshold value, $\gamma_{st} \approx 0$, may be the same situation as the $Q^2 \approx 8$ case. Another phase transition point is the usual phase transition point between the strong coupling region (the crumpled phase) and the intermediate region (the smooth phase). We expect that the phase transition at κ_2^c is continuous. This leads to the continuum limit of four-dimensional quantum gravity.

Now, let us take a look at the transition at κ_2^c in the case of $N_V = 1$. In order to investigate the phase transition, we observe at the exponents of the node susceptibility. The node susceptibility is defined in ref. [12] as follows:

$$\chi = \frac{1}{N_4} \left(\langle N_0^2 \rangle - \langle N_0 \rangle^2 \right). \quad (9)$$

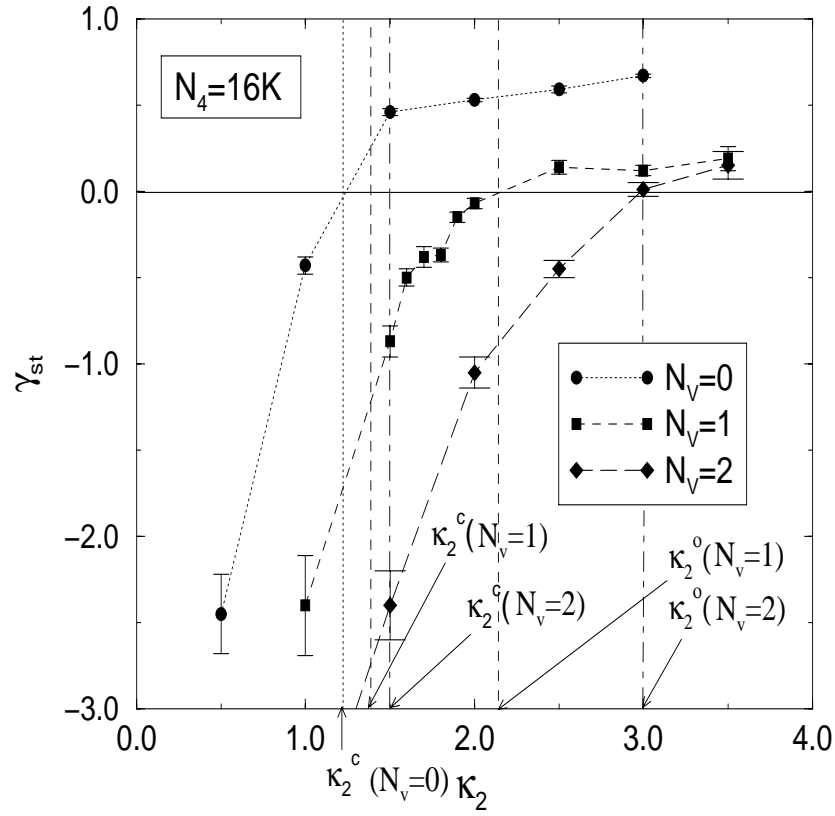


Figure 1: String susceptibility exponents (γ_{st}) plotted versus the coupling constants (κ_2) for $N_V = 0, 1$ and 2 .

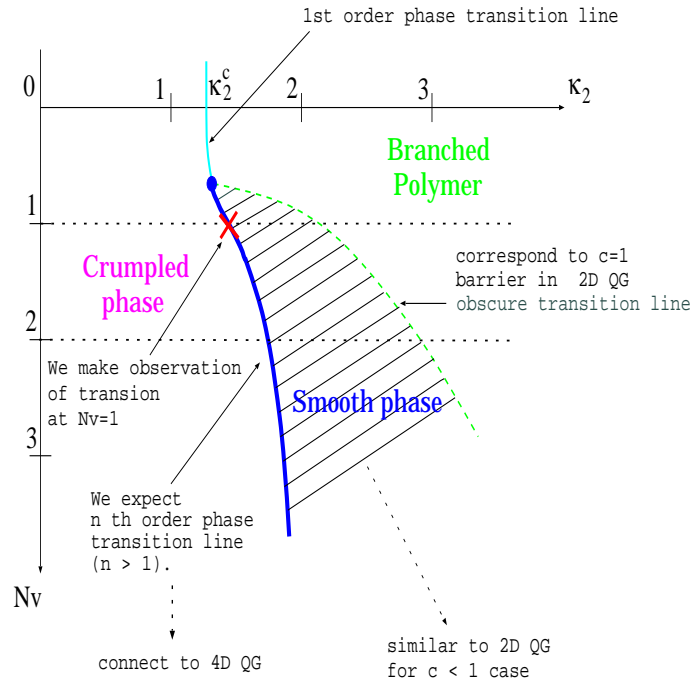


Figure 2: Schematic phase diagram of four-dimensional simplicial quantum gravity.

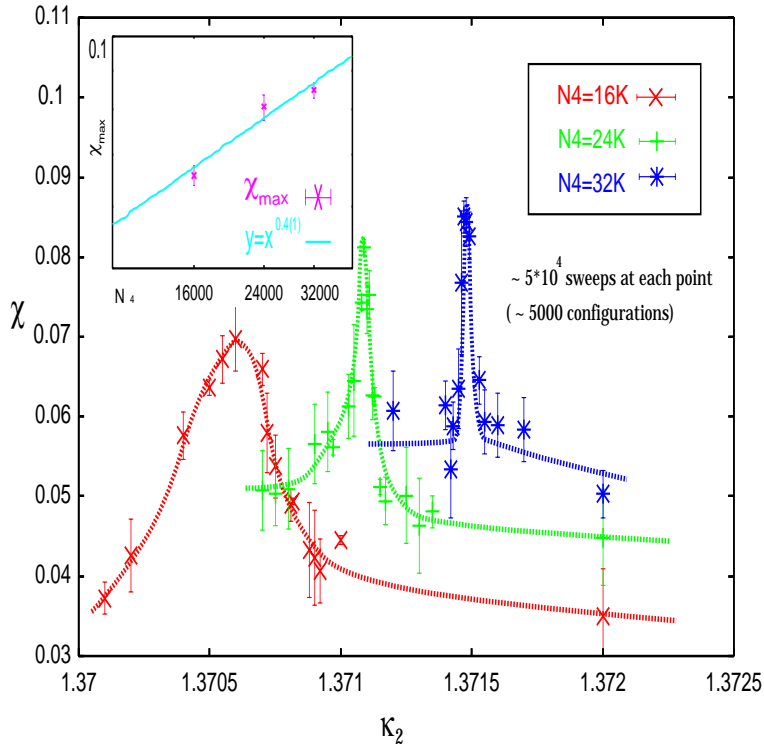


Figure 3: Node susceptibility (χ) plotted versus the coupling constant (κ_2) for $N_V = 1$.

In Fig.3, we plot the node susceptibility(χ) as a function of κ_2 with the volume $N_4 = 16K, 24K$ and $32K$, respectively. The node susceptibility (χ) has a peak value at the critical point (κ_2^c). We find the peak value in each size. The height and the width of the susceptibility peaks give the finite size scaling exponents of the phase transition. The peak value (χ_{max}) and the width of peak ($\delta\kappa_2$) grow as N_4 in power. The susceptibility exponents (Δ and Γ) are defined by [12]:

$$\chi_{max} \propto N_4^\Delta, \quad (\delta\kappa_2 \propto N_4^{-\Gamma}). \quad (10)$$

From the numerical results (Fig.3), we obtain the susceptibility exponents: $\Delta = 0.4(1)$, $\Gamma \sim 0.5(3)$. These values are apparently smaller than 1, though they are 1 [12] in the pure gravity case. This numerical results show that simplicial quantum gravity coupled to a vector field has the different type of phase transition from simplicial quantum gravity in the pure gravity case.

In Fig.4, we show a histogram of N_0 for a size of $N_4 = 32K$ near to the critical point ($\kappa_2^c = 1.37147(1)$). In the pure gravity case, previously, a double peak structure has been found [11, 12]. This structure relates to the latent energy. The fact that the phase transition is first order is shown. However, by adding the vector field, the double peak structure disappears. We thus conclude that the phase transition between the crumpled phase and the smooth phase may be continuous, not first order.

4 Scaling property of four-dimensional simplicial quantum gravity

Recent numerical results obtained by dynamical triangulation in four space-time dimensions suggest the existence of the scaling behavior, for example, the MINBU distribution [23], correlation functions [25] and the boundary volume distributions [10]. If we assume the existence of

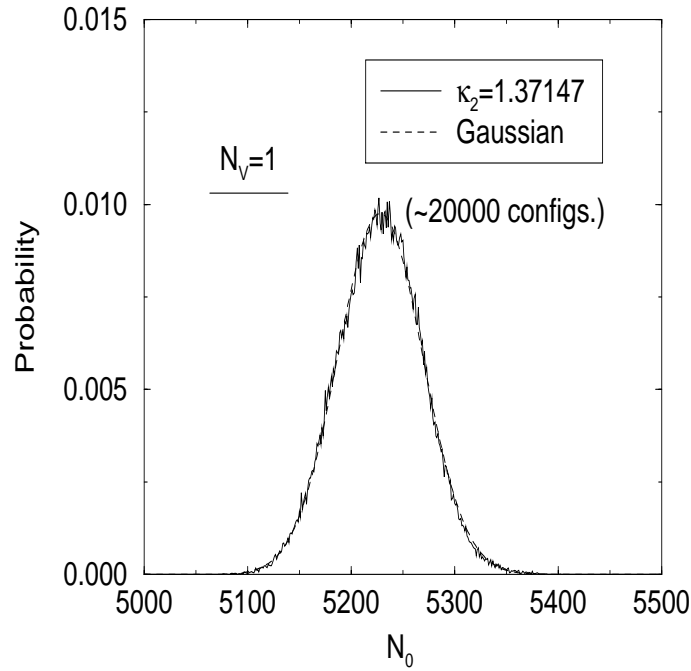


Figure 4: Histogram of node (N_0) near to the critical point ($\kappa_2^c = 1.37147(1)$) for $N_V = 1$ with $N_4 = 32K$. The number of configurations is 10,000.

a correct continuum limit, the scaling property of quantum geometry gives important information about the continuum theory, because some scaling properties are related to the universality of the theory. One of the interesting observes is the fractal dimension (Hausdorff dimension) (d_H),

$$d_H = \frac{d \ln V^{(4)}(D)}{d \ln D}, \quad (V^{(4)} \sim D^{d_H}). \quad (11)$$

This is based on studying the behavior of the volume $V^{(4)}(D)$, the number of simplices, within a geodesic distance D . The geodesic distance D is defined as the shortest distance between two simplices through the center of the simplices. In the pure gravity case, the Hausdorff dimension shows a different behavior in each phase. In the crumpled phase, the Hausdorff dimension diverges ($d_H \rightarrow \infty$). The Hausdorff dimension rises very steeply to a large value below the transition, while it falls very rapidly to a branched polymer, $d_H = 2$, above the transition. However, in the case of gravity coupled to vector fields, each of three phases shows a different behavior for the Hausdorff dimension. In the crumpled phase and the branched polymer phase, the behavior for the Hausdorff dimension is similar to that of pure gravity. Furthermore, we find that the smooth phase is the intermediate region with $2 < d_H \sim 4$. For the smooth phase, the value of the Hausdorff dimension (d_H) is changed to smooth, as compared with the pure gravity case. This fact supports the results of a finite size analysis. Especially, at the critical point, we observe the Hausdorff dimension, $d_H = 4.6(2)$, with $N_4 = 32K$.

Next, let us discuss the scaling structure of four-dimensional DT mfd, at the focusing on the scaling structure of the boundaries in four-dimensional Euclidean space-time using the concept of geodesic distances. We consider that the scaling structure of the boundaries has more informations than the Hausdorff dimension, and that it is the way of directly searching for the structure of quantum geometry. In the pure gravity case, the scaling property of the boundary volume is discussed in ref. [10]. As in the previous analysis [10], we assume that

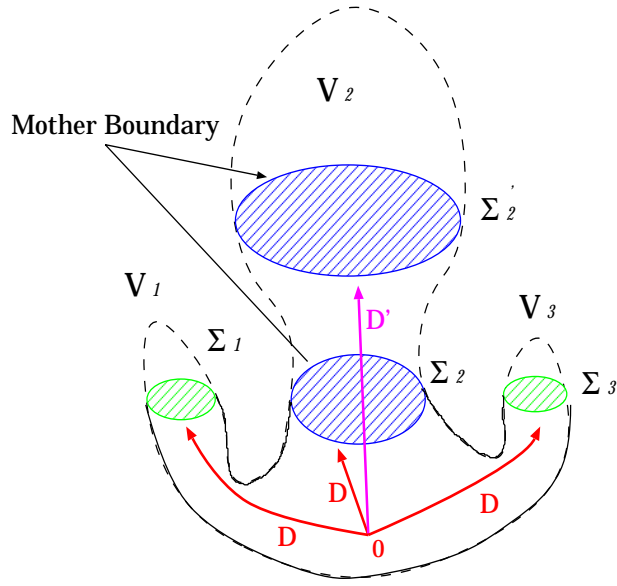


Figure 5: Schematic picture of the boundaries (Σ_1, Σ_2 and Σ_3) at distance D and a boundary (Σ_2') at distance D' in four-dimensional Euclidean space with S^4 topology. The mother boundary is defined as a boundary (Σ_2) with the largest tip volume (V_2); the other boundaries are defined as baby boundaries.

the boundary volume distribution ($\rho(x, D)$) is a function of a scaling variable in analogy with the loop length distribution function in the two-dimensional model. Fortunately, in the two-dimensional model, the loop length distribution function has been calculated analytically [4]. The loop length distribution function, $\rho(x = L/D^2, D)$, which gives the probability of the boundary loop with the loop length (L) within a geodesic distance (D), is given as a function of the scaling variable $x = L/D^2$:

$$\rho(x = L/D^2, D) = \frac{3}{7\sqrt{\pi}} \frac{1}{D^2} \left(x^{-3/2} + \frac{1}{2} x^{-5/2} + \frac{14}{3} x^{1/2} \right) e^{-x}. \quad (12)$$

This distribution function consists of two different types of distributions. The first two term of eq.(12) represent the baby loops that the universe has a small boundary volume; the last term represents the mother loop that the universe has a large boundary volume. The distribution function, $\rho(x = L/D^2, D)$ of eq.(12), satisfies the scaling relation under rescaling ($D \rightarrow D' = \sqrt{\lambda}D, L \rightarrow L' = \lambda D$):

$$\rho(L, D) = \lambda^{-1} \rho(L', D'). \quad (13)$$

In the four-dimensional model, unfortunately, a similar scaling relation about the boundary volume is not yet known. We thus assume the distribution function, $\rho(x = V/D^\alpha)$, of the volume of the boundary (V) within the geodesic distance (D) in four-dimensional dynamical triangulated manifold. This is a function of the scaling variable, $x = V/D^\alpha$, with scaling parameter (α) in analogy with the two-dimensional model. In Fig.5, we show a schematic picture of our boundary analysis.

First, we consider the scaling structures of these three phases: a crumpled phase, a smooth phase and a branched polymer phase [20]. Actually, in the smooth phase we observe that the distribution (ρ) becomes fractal in the sense that the sections of the manifold at different distances from a given 4-simplex look exactly the same after a proper rescaling of the boundary

volume. Furthermore, the shape of this scaling function is very similar to that of the two-dimensional case [4, 3]. The best account for this excellent agreement in the four-dimensional case can be found in the dominance of a conformal mode and a fractal property. We have also investigated the boundary volume distribution in the crumpled phase and the branched polymer phase. It seems reasonable to suppose that this new smooth phase has a similar fractal structure to that of the two-dimensional fractal surface, and that there is a possibility of taking a continuum limit in the phase. We have also investigated the boundary volume distribution in both the crumpled phase and the branched polymer phase. In the crumpled phase we find that one mother universe is dominant, while in the branched polymer phase there is no evidence for the existence of a mother universe.

Next, let us discuss the relation between the scaling parameter (α) and the Hausdorff dimension (d_H). The expectation value of the boundary three-dimensional volume appearing at distance D has been introduced in ref. [10]:

$$\langle V^{(3)} \rangle = \frac{1}{N} \int_{v_0}^{\infty} dV V \rho(x = V/D^\alpha, D), \quad (14)$$

where v_0 denotes the UV cut-off of the boundary volume and N is the normalization factor. If the boundary volume has the scaling property with the universal distribution ($\rho(x, D)$) and $v_0 \rightarrow 0$,

$$\langle V^{(3)} \rangle \sim D^\alpha. \quad (15)$$

Then, we should obtain a finite fractal dimension,

$$d_f = \alpha + 1, \quad (16)$$

with the fractal dimension d_f . We measure the volume of the mother boundary as a function of D . The mother boundary is defined by the boundary having the largest tip volume. In Fig.6, we plot the mother boundary volume $\langle V^{(3)} \rangle$ with the size of $N_4 = 32K$ at the critical point. As a result, the mother boundary volume shows a scaling, and we obtain the scaling parameter ($\alpha = 3.7(5)$). Then, we can estimate the fractal dimension ($d_f = 4.7(5)$). On the other hand, we measured the Hausdorff dimension, which results in $d_H = 4.6(2)$. Both results are consistent ($d_f \approx d_H$). We thus expect that the boundary volume has a scaling property in the sense that the manifold at different distances from a given 4-simplex looks exactly the same after a proper rescaling of the boundary volume.

5 Modified balls-in-boxes model

In this section, we discuss a scenario for the phase structure and the phase transition in four-dimensional simplicial quantum gravity coupled to matter fields. In the pure gravity case, the first order phase transition is described by the balls-in-boxes model [16–18]. This model is considered to be a simple mean field model about simplicial quantum gravity. It describes a fixed number N balls distributed into a variable M of boxes. The partition function is given by

$$Z_{M,N} = \sum_{q_1, \dots, q_M} p(q_1) \cdots p(q_M) \delta_{q_1 + \dots + q_M, N}, \quad (17)$$

where q_i is the number of balls in box i , and $p(q)$ denotes a weight which depends on only the number of balls (q) in the box. The original balls-in-boxes model has the following correspondence:

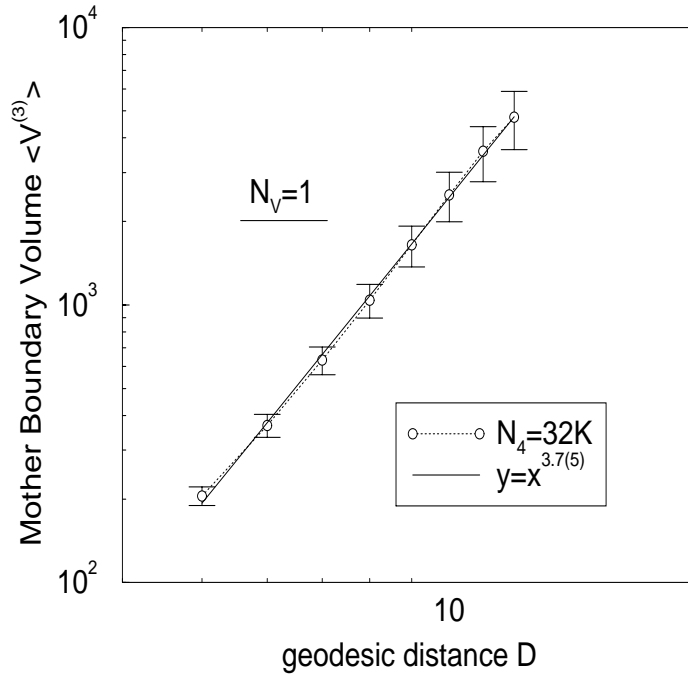


Figure 6: Power fit to the mother boundary three-dimensional volume $\langle V^{(3)} \rangle$ at the critical point ($\kappa_2^c = 1.37147(1)$) for $N_V = 1$ with $N_4 = 32K$.

Vertex \leftrightarrow Box and Simplex \leftrightarrow Ball.

Therefore, q_i represents the vertex order. The original model has been analyzed in the case of power like weights: $p(q) = q^{-\beta}$ (when β is a parameter for a weight). The system has two distinct phases: a crumpled phase and an elongated phase, and it has a discontinuous phase transition for $\beta > 2$. It describes the phase structure about four-dimensional pure simplicial quantum gravity. However, this model describes the continuous phase transition at the $\beta < 2$ [18]. We have noticed this fact, which gives one of the motivations for the modified model of simplicial quantum gravity. In the model for simplicial quantum gravity, we can consider that parameter β is related to the measure term. If we consider the conformal matter fields (as the conformal charge), parameter β has the same origin as the conformal anomaly, because they are related to the measure term. The gauge field gives rise to a modified measure factor [21],

$$M_t = \prod_t o(t)^\beta. \quad (18)$$

This model shows three distinct phases: a crumpled phase, a crinkled phase (correspond to a smooth phase) and a branched polymer phase. It is consistent with our results. The natural correspondence between our phase structure and that in Ref. [21] is as follows: the transition between the crumpled phase and the branched polymer phase is discontinuous (probably, first-order phase transition), the transition between the crumpled phase and the crinkled phase is continuous, and the transition between the crinkled phase and the branched polymer phase is obscure.

Thus, in order to investigate the phase structure of adding matter fields, we exchange the relation of “vertices-simplices” into that of “triangle-simplices”:

Triangle \leftrightarrow Box and Simplex \leftrightarrow Ball.

We thus introduce the triangle order ($o(t)$) instead of the vertex order (q). This modification causes the partition function to give rise a constant shift for the constraint:

$$\sum_{i:vertex} q_i = 5N_4, \quad (19)$$

$$\sum_{t:triangle} o(t) = 10N_4. \quad (20)$$

We analyzed the modified balls-in-boxes model with using the same method of the both original models.

We thus considered the case of simplicial quantum gravity coupled to matter fields. The partition function is

$$Z_N(\kappa, \beta, N_{matter}) = \sum_m^{M_{max}} e^{\kappa m} \sum_{q_1, \dots, q_m} p(q_1) \cdots p(q_m) \delta_{q_1 + \dots + q_m, N} (Z_{matter}(m))^{N_{matter}}. \quad (21)$$

Now we discuss the following two cases: (a) the case of coupling to N_B copies of the boson fields and (b) the case of coupling to N_V copies of the vector fields. (a) In the first case, the matter action is

$$S_{boson} = \sum_{(ij)} (\phi_i - \phi_j)^2. \quad (22)$$

We discuss the effect from the matter. From the perturbation, the Gaussian boson matter gives

$$Z_{boson} = \sum_{triangles} Constant. \quad (23)$$

We can estimate the effect to the weight of one-boxes:

$$p(q) \rightarrow C \cdot p(q). \quad (24)$$

The Gaussian bosons give the effect of multiplication by a constant. From a mean fields analysis, the phase diagram will be changed. However, it is known that the phase diagram is not changed by only a few Gaussian boson fields, according to numerical analysis [26]. We consider that the multiplication by a constant is not enough to change the phase diagram for “real” simplicial quantum gravity, because of the too small effect of multiplication by a constant. (b) In the second case, we consider adding the Gaussian $U(1)$ gauge matter. If we use the plaquette action for the gauge fields, the partition function is written as

$$Z_{gauge} = \sum_{triangle} f(o(t)), \quad (25)$$

where f is a function of the triangle order ($o(t)$). From the perturbation about the gauge interaction, $f(o(t)) = C_0 + C_1 o(t)$. Then, the partition function is replaced, as follows:

$$Z_{gauge} \sim \sum_t o(t). \quad (26)$$

This is just the modified measure. We estimate the effect from the gauge matter fields as

$$\beta \rightarrow (\beta - 1). \quad (27)$$

That is to say, the Gaussian gauge matter fields lower β .

6 Summary and Discussion

We have investigated the phase structure and the phase transition with a model of four-dimensional simplicial quantum gravity coupled to $U(1)$ gauge fields.

The results of our study are summarized by the schematic phase diagram in Fig.2. We checked this phase diagram in the case of $N_V = 1, 2, 3$ at a volume of $N_4 = 16K$. We found three phases in this parameter space: a crumpled phase ($\gamma_{st} \rightarrow \infty$, $d_f \rightarrow \infty$), a smooth phase ($\gamma_{st} < 0$, $2 < d_f \sim 4$, $N_0/N_4 < 0.25$) and a branched polymer phase ($\gamma_{st} > 0$, $d_f = 2$, $N_0/N_4 = 0.25$). The thin line denotes a discontinuous phase-transition line which is known in pure gravity; moreover, the thick line denotes a smooth phase-transition line. As contrasted with the phase diagram of pure gravity, the phase diagram means richer structures. In the crumpled phase one can find singular sub-simplices, for example, vertex order concentration and link order concentration. The smooth phase is defined as a region between the critical point (κ_2^c) and the obscure phase transition point (κ_2^o). We observe the negative string susceptibility in this region. Then, with $N_4 = 16K$ ($\kappa_2 = 1.7$) and $N_V = 1$, we obtained $\gamma_{st} = -0.38(5)$, $d_f = 2.8(5)$ and a good scaling relation of the boundary volume distributions. We consider that the scaling structure of this smooth phase is similar to that of a two-dimensional random (fractal) surface. This smooth phase is slowly broken to a branched polymer phase that has no mother structure at the obscure phase transition (κ_2^o). We obtained an obscure-transition line (a broken line in Fig.2); therefore, we suggest that the obscure-transition corresponds to the $c = 1$ barrier in two-dimensional quantum gravity.

As for the phase transition at the critical point ($\kappa_2 = \kappa_2^c$), we showed the finite size scaling at the critical point and the histogram of the node. We calculated some finite size scaling exponents, and showed that the value of these exponents is smaller than 1. It has been discussed [12] that the value of 1 is expected to be a first order phase transition for the results of pure gravity. However, in the case of adding one gauge matter field ($N_V = 1$), the numerical results show that the phase transition is smooth, in the contrast to pure gravity, and then that a double peak structure of the node histogram disappears.

Furthermore, in order to investigate the property of quantum geometry and to discuss the universality of the manifold, we considered the scaling property of the boundary volume. In the smooth phase with $N_4 = 16K$ ($\kappa_2 = 1.7$) and $N_V = 1$, we obtained $d_f = 2.8(5)$ and a good scaling relation of the boundary volume distributions. The scaling structure of this smooth phase is similar to that of a two-dimensional random (fractal) surface. This suggests the existence of a new smooth phase in four-dimensional simplicial gravity. The other two phases have a similar scaling property to that of pure gravity. We have shown the scaling property of the mother boundary, where the scaling parameter is consistent with the Hausdorff dimension. Then, the boundary volume has a scaling property with the scaling variable ($x = V/D^{d_f-1}$). We expect that the boundaries have a fractal structure and universality of the scaling relations. We also discussed the modification of the balls-in-boxes model. The role of a vertex is exchanged into a triangle. This clarifies the relation between the measure factor of the numerical model and that of the analytical mean field model.

We expect the existence of genuine four-dimensional quantum gravity on the critical point (κ_2^c) with the vector fields. Our recent investigations will give further evidence for the existence of an ultraviolet fixed point of the quantum field theory of gravity.

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